Journal of Statistical Physics, Vol. 65, Nos. 3/4, 1991

Two-Dimensional Random–Random Walks: Dynamical Exponents in a Quenched Directed Model

Claude Aslangul,[1] Marc Barthélémy,[1] Noëlle Pottier,[1] and Daniel Saint-James[2,3]

Received April 19, 1991; final June 20, 1991

This paper presents a study of the dynamics of a particle undergoing a directed random walk in a two-dimensional disordered square lattice. We derive the asymptotical behaviors of the coordinate and of the mean square displacement. All the dynamical exponents are calculated both in the normal and the anomalous regimes. It is shown that, as contrasted to the one-dimensional case, the so-called quenched and annealed diffusion "constants" indeed coincide.

KEY WORDS: Fluctuation phenomena; random processes and Brownian motion.

1. INTRODUCTION

Random–random walk generally designates Brownian motion of a "particle" in a medium having a feature which varies randomly from one point to another. The term medium is to be taken in a wide sense: it can be the ordinary physical space, or any other parameter space such as an energy space, for instance.[(1)] In this latter case, the walk pictures the motion of a system from one energy state to another. On a general level, Brownian motion in the viscous limit can be captured by an ordinary master equation giving the time evolution of the probabilities to have a definite value for some dynamical parameter (coordinate, population of a given species,...).

[1] Groupe de Physique des Solides, Laboratoire associé au CNRS (UA no. 17) et aux Universités Paris VII et Paris VI, 75251 Paris Cedex 05, France.
[2] Laboratoire de Physique Statistique, Collège de France, 75231 Paris Cedex 05, France.
[3] Also at Université Paris VII, 75251 Paris Cedex 05, France.

0022-4715/91/1100-0673$06.50/0

In what follows, for definiteness, we shall use the picture of a particle moving on a lattice. In such a case, the main relevant dynamical parameters are the position of the particle and its mean-square displacement; these quantities are found by achieving the proper averages over the probability distribution as given by the master equation and, when required, over all the disorder configurations. Overbarring will denote an expectation value with respect to the probabilities, while brackets $\langle \cdots \rangle$ mean a disorder average.

In one dimension ($d = 1$), the symmetric walk, in which forward and backward jumps are given the same transition rate, was considered some years ago by Alexander *et al.*[2] Later, in view of discussing the transport properties of disordered materials, the asymmetric case was introduced, allowing the possibility for a drift to occur, in addition to the unavoidable diffusive process. It was shown by Derrida[3] that, according to the nature of the disorder, anomalous dynamical behaviors can arise in the sense that, provided that the disorder is strong enough, the mean coordinate is not simply proportional to the time t, but increases slower than t, whereas the mean square displacement can grow either faster or slower than t. Generally speaking, the asymptotic dynamics is characterized by exponents which are simply equal to one in the standard regime, and have to be explicitly found in anomalous cases. It is now common to speak of "dynamical phases" as soon as at least one exponent is not standard. A detailed account of this problem was recently published.[4] It is worth noting that, within the usual equivalence between dynamics at large times for a random walk and critical phenomena,[4] anomalous regimes correspond to relevant deviations from mean-field behavior. Many previous treatments, especially for $d > 1$, have used mean-field approximations, while others go beyond with renormalization group (RG) methods. Clearly, an exact treatment is of valuable interest. Such an exact solution is presented here for the special case of a directed walk at $d \geqslant 2$.

The general walk is indeed a very hard problem, even for $d = 1$, for which only a few results can be explicitly derived, although many insights can be obtained with the use of semiquantitative arguments. On the other hand, a simplified version of the model turns out to be fully soluble, in which, for each direction in space, only jumps in one way are allowed.[5] This directed walk was analyzed in refs. 6 and 7 in the one-dimensional case; in these papers, we explicitly derived all the dynamical exponents and discussed various self-averaging properties of the model. The aim of the present paper is to show that the solution for $d \geqslant 2$ can also be found and to provide an exact analysis of the corresponding asymptotic dynamics.

The directed walk is of interest *per se*. First of all, it pictures a dynamical process in which the variable of interest can only increase,[1] as is the case, for instance, when particles are deposited on a random substrate as a result of a diffusive process in its vicinity; there, the dynamical variable of interest is the number of particles adsorbed. On the other hand, general physical arguments tend to show that the general both-way one-dimensional walk is, on large time and space scales, in a sense equivalent to a directed walk on a renormalized lattice[4,8]; however, no genuine algebraic proof is available. The status of such an equivalence for higher-dimensional models is an open question. As shown below, anomalous regimes will emerge in any dimension for the directed walk, although their domain of existence keeps reducing when d increases. On the other hand, RG methods establish that, for a vanishing average bias, anomalous phases are absent for $d>2$ (see ref. 4, Section 4.3); in addition, as shown by Bricmont and Kupiainen,[9] ordinary diffusion occurs in the presence of a bias when $d>2$ for a wide class of weak local random perturbations. There is no contradiction, in the sense that the directed walk which would correspond to a general walk, if an equivalence would hold, makes sense only within some high-field assumption, in which case standard regimes do occur in both models. Moreover, the dynamics in the presence of strong disorder is not well known for the general walk and this is also a case in which the basic features of each model (loops or no loops) may probably be capable of generating different behaviors.

Anyway, both-way walks and directed walks essentially differ in the fact that in the former case the particle can visit each site an arbitrary number of times, whereas, in the latter, a given site can be traveled through at most once. It can thus be expected that the throw-back of fluctuations for $d>1$, where loops do exist, is more effective for a general walk than in a directed walk.

2. BASIC EQUATIONS

Calling $p_{nm}(t)$ the probability for the particle to be at the site with coordinates (n, m) at time t, we have for the master equation for the present directed model on a rectangular lattice (see Fig. 1)

$$\frac{dp_{nm}(t)}{dt} = W_{x,n-1m}\, p_{n-1m}(t) + W_{y,n,m-1}\, p_{nm-1}(t) - (W_{x,nm} + W_{y,nm})\, p_{nm}(t) \tag{1}$$

For any function $f(t)$, we now define its Laplace transform $F(z)$ in the usual way:

$$F(z) \equiv L[f(t)] = \int_0^{+\infty} dt\, e^{-zt} f(t) \tag{2}$$

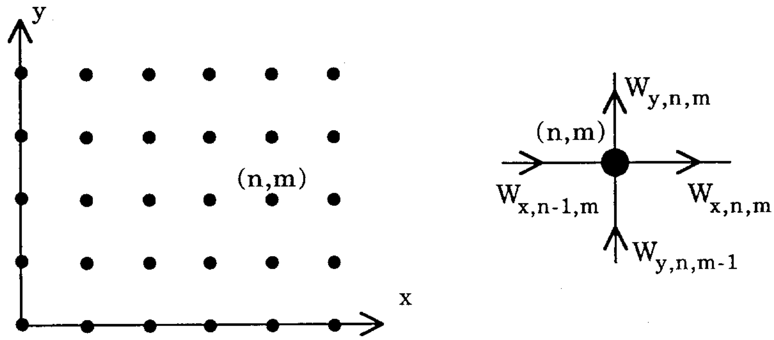

Fig. 1. Schematic picture of the lattice and labeling of the random transition rates.

In all the following, the various Laplace transforms are taken with functions which are all bounded at infinity, so that the definition (2) is valid for any complex number z having a positive real part.

By taking the Laplace transform of Eq. (1) with the initial condition $p_{n,m}(t=0)=\delta_{n0}\delta_{m0}$, one readily obtains

$$zP_{nm}(z)-\delta_{n0}\delta_{m0}=W_{x,n-1m}P_{n-1m}(z)+W_{y,nm-1}P_{nm-1}(z)$$
$$-(W_{x,nm}+W_{y,nm})\,P_{nm}(z) \tag{3}$$

The transition rates $W_{u,nm}$ $(u=x,\ y)$ are assumed to be positive independent random variables, all chosen in two given densities, $\rho_x(W)$ and $\rho_y(W)$, one for each space direction; thus, the disorder average of any function of the W's is independent of the position in space, a fact which restores a translational symmetry in the mean. With the aim to investigate the whole range from strong to weak disorder in a simple way, and keeping in mind that the asymptotic dynamics at large times is essentially governed by the behavior of the ρ's near $W=0$, we make the following choice:

$$\rho_u(W)=\frac{1}{\Gamma(\mu_u)}\,W_c^{-\mu_u}W^{\mu_u-1}e^{-W/W_c} \qquad (u=x,\ y,\quad \mu_u>0) \tag{4}$$

where Γ is the Euler function of the first kind.

Thus, each ρ_u is a gamma distribution characterized by the parameter μ_u and with the common cutoff W_c; the precise W dependence of the cutoff function (here of the exponential form) is expected to have no bearing on the dynamics at large times, except for numerical prefactors. μ_u is constrained to be strictly positive and is a measure of the surrounding disorder; strong disorder is realized when μ_u is small: in particular, when $\mu_u<1$, the density (4) diverges near the origin, allowing for a high probability to find a nearly-broken link. On the other hand, infinite μ values restore the pure (ordered) lattice; the limiting process is precisely the following:

$$(\mu_u \to +\infty,\ W_c \to 0,\ \mu_u W_c = W_u) \Rightarrow \rho_u(W) \to \delta(W - W_u) \quad (u = x, y) \tag{5}$$

The gamma density has a useful property which will be used here and there in the following: it is stable under the addition of independent variables; otherwise stated, if the independent random variables W_x and W_y are distributed according to ρ_x and ρ_y, the sum $W = W_x + W_y$ is also distributed according a gamma law:

$$\rho(W) = \frac{1}{\Gamma(\mu)} W_c^{-\mu} W^{\mu-1} e^{-W/W_c} \qquad (\mu = \mu_x + \mu_y) \tag{6}$$

having the parameter μ simply equal to the sum $\mu_x + \mu_y$.

The subsequent analysis essentially aims at providing the dynamical exponents for the two first moments of the particle position. We thus first write for completeness the so-called thermal averages:

$$\overline{x^r(t)\, y^s(t)} = \sum_{n=0}^{n=+\infty} \sum_{m=0}^{m=+\infty} n^r m^s p_{nm}(t) \tag{7}$$

which, for $(r, s) = (1, 0)$ or $(0, 1)$, describe the average position in the lattice plane for a given (quenched) disorder configuration. The Laplace transform of these latter moments will be denoted by the corresponding subscript and without overbarring in order to simplify the notations:

$$x_r(z) \equiv L[\overline{x^r(t)}], \qquad y_s(z) \equiv L[\overline{y^s(t)}], \qquad (x_r y_s)(z) = L[\overline{x^r(t)\, y^s(t)}] \tag{8}$$

The mean square dispersions are thus obtained by forming the combinations:

$$\begin{aligned} \overline{\delta_{xx}(t)} &= \overline{x^2(t)} - \overline{x(t)}^2 \\ \overline{\delta_{xy}(t)} &= \overline{x(t)\, y(t)} - \overline{x(t)}\ \overline{y(t)} \\ \overline{\delta_{yy}(t)} &= \overline{y^2(t)} - \overline{y(t)}^2 \end{aligned} \tag{9}$$

For future use, we introduce the moment-generating function $\Phi(\phi, \psi; z)$:

$$\Phi(\phi, \psi; z) = \sum_{n=0}^{n=+\infty} \sum_{m=0}^{m=+\infty} e^{in\phi} e^{im\psi} P_{nm}(z) \tag{10}$$

which, by the proper derivations at $\phi = 0$ and/or $\psi = 0$, directly gives the Laplace transforms of all the moments defined in Eq. (7).

All the previous averages are related to a given sampling of the lattice and are the primary quantities of physical interest when one looks for a detailed description of the dynamics in a given sample of a disordered material. Their full calculation is a rather formidable task. The problem can also be tackled in a different and easier way. In a first step, one only tries to obtain the dynamics for *disorder-averaged* moments ($\langle \overline{x^r(t)} \rangle$, and so on), which are representative of the motion averaged over an infinite ensemble of independent samples. In a second step, the sample-to-sample possible fluctuations are analyzed. We followed this procedure in refs. 6 and 7 for $d = 1$. In the present two-dimensional analysis, we shall provide results for the disorder averages of the thermal expectation values. As explained below, some of these results allow one to draw some conclusions about the self-averaging properties of the thermal expectation value of the coordinate.

As for the disorder-averaged mean square displacements, it is customary to distinguish the following two kinds:

(i) The quenched mean square displacements $\delta_{uv,Q}(t) = \langle \overline{u(t)\, v(t)} \rangle - \langle \overline{u(t)}\, \overline{v(t)} \rangle$, which indeed coincide with the physical dispersions averaged over disorder, relevant for an ensemble of quenched lattices.

(ii) The annealed mean square displacements $\delta_{uv,A}(t) = \langle \overline{u(t)\, v(t)} \rangle - \langle \overline{u(t)} \rangle \langle \overline{v(t)} \rangle$, which differ from the previous ones by the insertion of an additional disorder average in the last term. Clearly, this latter definition *a priori* discards specific sample-to-sample fluctuations, so that, in general, one expects $\delta_{uv,Q} \neq \delta_{uv,A}$. More precisely, one easily sees that the following inequality certainly holds for the diagonal elements:

$$\delta_{uu,A}(t) \geqslant \delta_{uu,Q}(t) \tag{11}$$

The connection between these two mean square displacements is in some way related to the self-averaging properties of $\overline{x(t)}$. Namely, when the quenched and annealed mean square displacements coincide, then $\overline{x(t)}$ is self-averaging, but the converse is not necessarily true. Indeed, the self-averaging property statement only relates to the dominant contribution to $\overline{x(t)}$, whereas the *subdominant* terms can by themselves build up the difference. This point is worth discussing in some detail.

Let us assume for definiteness that, at large times, the two first moments of the coordinate $\overline{x(t)}$ for a given lattice are such that

$$\overline{x(t)} = at^{\alpha} + \xi(t), \qquad \langle \overline{x^2(t)} \rangle \approx At^{2\alpha} + Bt^{\beta}$$

where $\xi(t)$ is a noisy correction and where $\beta < 2\alpha$. From this assumption, one readily obtains for the difference

$$\delta_{xx,A}(t) - \delta_{xx,Q}(t) = (\langle a^2 \rangle - \langle a \rangle^2)\, t^{2\alpha} + 2(\langle a\xi(t) \rangle - \langle a \rangle \langle \xi(t) \rangle)\, t^{\alpha} + \langle \xi^2(t) \rangle - \langle \xi(t) \rangle^2$$

When $\delta_{xx,A} \sim \delta_{xx,Q}$, this means that the first term vanishes (and that the variance of the noise ξ does increase slower than $t^{2\alpha}$), which in turn implies that the prefactor a in $\overline{x(t)}$ displays no sample-to-sample fluctuations, i.e., $\overline{x(t)}$ is self-averaging. On the other hand, when this last quantity is self-averaging, this only implies that the dominant term in $\overline{x(t)}$ is not random; thus, one is left with

$$\delta_{xx,A}(t) - \delta_{xx,Q}(t) = \langle \xi^2(t) \rangle - \langle \xi(t) \rangle^2$$

which does not allow one to conclude that $\delta_{xx,A} \sim \delta_{xx,Q}$ when there is no other available information. In particular, for a standard regime $(\alpha = 1, \beta = 1)$, a self-averaging $\overline{x(t)}$ together with a common centered Gaussian noise $\xi(t)$ leads to

$$\delta_{xx,A}(t) - \delta_{xx,Q}(t) = \text{const} \times t$$

at large times. This last relation immediately shows that the quenched and annealed diffusion constants $D_{xx,A}$ and $D_{xx,Q}$ are not the same since $\delta_{xx,A}$ and $\delta_{xx,Q}$ both asymptotically behave like t. It thus turns out that the coincidence between $\delta_{xx,A}$ and $\delta_{xx,Q}$ is basically related to the self-averaging property, not of $\overline{x(t)}$ itself, but of the subdominant correction for a given sample. We shall demonstrate below that this coincidence holds in all dynamical phases for the two-dimensional directed walk, a fact which ensures that $\overline{x(t)}$ is here self-averaging, as contrasted with the one-dimensional case.[6, 7] A discussion of the difference between D_A and D_Q can be found in refs. 4 and 10.

3. ASYMPTOTICS FOR THE VELOCITY

The calculation of the disorder-averaged moments proceeds through the determination of the averaged generating function $\langle \Phi(\phi, \psi; z) \rangle$ defined in Eq. (10). Any averaged probability $\langle P_{nm}(z) \rangle$ can be found as follows. First, one notes that P_{nm} can be expressed as

$$P_{nm}(z) = G_{nm}(z)(W_{x,n-1m} P_{n-1m} + W_{y,nm-1} P_{nm-1}), \qquad P_{00}(z) = G_{00}(z) \tag{12}$$

where

$$G_{nm}(z)=\frac{1}{z+W_{x,nm}+W_{y,nm}}$$

All the paths going from the site (0, 0) to the site (n, m) have n steps to the right and m steps upward. To every elementary jump is associated an "amplitude" $A_{u,nm}$ given by

$$A_{u,nm}(z)=\frac{W_{u,nm}}{z+W_{x,nm}+W_{y,nm}}\equiv W_{u,nm}G_{nm}(z)$$

Thus, the probability $P_{nm}(z)$ is the sum the contributions of all these paths, each of them containing n factors A_x and m factors A_y. Due to the fact that the W's are uncorrelated and that they are all chosen with the same densities ρ_u, it is readily seen that the following equality holds:

$$\begin{aligned}\langle P_{nm}(z)\rangle &= C^n_{n+m}\left\langle\frac{1}{z+W_x+W_y}\right\rangle\left\langle\frac{W_x}{z+W_x+W_y}\right\rangle^n\left\langle\frac{W_y}{z+W_x+W_y}\right\rangle^m\\ &\equiv C^n_{n+m}R(z)\langle A_x(z)\rangle^n\langle A_y(z)\rangle^m \qquad \left(C^p_n=\frac{n!}{p!\,(n-p)!}\right)\end{aligned} \tag{13}$$

where all the n, m indices have been dropped in the W's since they are unnecessary when such quantities appear within an average taken with the ρ_u's. The quantity $R(z)$, which will play a central role in the following, is the Stieltjes transform of $\rho(W)$ [see Eq. (6)], defined as

$$R(z)=\left\langle\frac{1}{z+W_x+W_y}\right\rangle=\int_0^{+\infty}\frac{dW}{z+W}\rho(W) \tag{14}$$

By its very definition, the function $R(z)$ has a cut extending over the whole real negative axis. Using Eq. (13), one can easily sum the series in Eq. (10), which yields

$$\langle\Phi(\phi,\psi;z)\rangle=R(z)[1-e^{i\phi}\langle A_x(z)\rangle-e^{i\psi}\langle A_y(z)\rangle]^{-1} \tag{15}$$

Now, by calculating the two derivatives $(i^{-1}\,\partial/\partial\phi)_{\phi=\psi=0}$ and $(i^{-1}\,\partial/\partial\psi)_{\phi=\psi=0}$, one gets the exact expressions

$$\langle x_1(z)\rangle=z^{-2}\frac{\langle A_x(z)\rangle}{R(z)},\qquad \langle y_1(z)\rangle=z^{-2}\frac{\langle A_y(z)\rangle}{R(z)} \tag{16}$$

The asymptotic dynamics can be found by looking at the behavior at small z of these quantities. Since the $\langle A\rangle$'s are always finite, the asymptotic regime critically depends on the behavior of $R(z)$ near $z=0$. In view of a systematic subsequent use of this property, we now quote the following expansion, easily derived from Eq. (6), in which Z denotes the reduced variable z/W_c and where $\mu=\mu_x+\mu_y$:

$$R(Z)=W_c^{-1}\left[\Gamma(1-\mu)\,Z^{\mu-1}+\frac{1}{\mu-1}-\frac{Z}{(\mu-1)(\mu-2)}\right.$$
$$\left.+\frac{Z^2}{(\mu-1)(\mu-2)(\mu-3)}+\cdots\right] \tag{17}$$

When reordered with respect to the dominant powers, this expansion displays the usual fact that, when μ increases from 0_+, the actual relevance of the multivalued term $Z^{\mu-1}$ step by step decreases every time that μ crosses an integer value. From Eqs. (16) and (17), one now has $(\mu=\mu_x+\mu_y)$

$$\mu<1: \qquad \langle\overline{x(t)}\rangle\sim\mu_x\frac{\sin\pi\mu}{\pi\mu^2}(W_ct)^{\mu} \tag{18a}$$

$$\mu>1: \qquad \langle\overline{x(t)}\rangle\sim\mu_x\frac{\mu-1}{\mu}W_ct \tag{18b}$$

and analogous expressions for $\langle\overline{y(t)}\rangle$ with μ_y instead of μ_x. Generally speaking,

$$\langle\overline{x(t)}\rangle=V_x(\mu)(W_ct)^{\alpha(\mu)} \tag{19}$$

where the exponents as well as the transport coefficient $V_x(\mu)$ are summed up in Table I.

Table I. Exponents and Transport Coefficients for the Various Dynamical Regimes Found in the Text [see Eqs. (19) and (28)]

	α	β	$V_x(\mu)$	$D_{xx}(\mu)$
$0<\mu<1$	μ	2μ	$\frac{\mu_x}{\mu}\frac{\sin\pi\mu}{\pi\mu}$	$\left(\frac{\mu_x}{\mu}\frac{\sin\pi\mu}{\pi\mu}\right)^2\left\{\frac{[\Gamma(\mu+1)]^2}{\Gamma(2\mu+1)}-\frac{1}{2}\right\}$
$1<\mu<2$	1	$3-\mu$	$\frac{\mu_x}{\mu}(\mu-1)$	$\left(\frac{\mu_x}{\mu}\right)^2\frac{(\mu-1)^2}{(3-\mu)(2-\mu)}$
$\mu>2$	1	1	$\frac{\mu_x}{\mu}(\mu-1)$	$\frac{\mu_x}{\mu}\frac{(\mu-1)}{\mu-2}\left[\mu+2\left(\frac{\mu_x}{\mu}-1\right)\right]$

Thus, in the plane (μ_x, μ_y), the frontier of the anomalous region for the velocity is the line $\mu_x + \mu_y = 1$. For any μ, the disorder-averaged trajectory at large times tends toward the straight line with the slope μ_y/μ_x. The motion along this line is slower for $\mu < 1$ as compared to the normal regime. For $\mu > 1$, the drift motion is normal and displays the velocity

$$\mathbf{V}(\mu) = \left(\mu_x \frac{\mu - 1}{\mu}, \mu_y \frac{\mu - 1}{\mu} \right) \qquad (\mu = \mu_x + \mu_y > 1) \tag{20}$$

As already quoted, the pure (nondisordered) limit is recovered achieving the limiting procedure defined in Eq. (5). Using Eqs. (18b) and (20), it is thus readily seen that, for the pure case,

$$\overline{x(t)} = W_x t, \qquad \overline{y(t)} = W_y t$$

as it should.

It can be said that, for this two-dimensional motion, the anomalous regime is harder to encounter. Indeed, even if μ_y, for instance, is small, a finite velocity does exist provided that, at the same time, μ_x is large enough so that the sum $\mu_x + \mu_y$ is greater than one. Globally, the motions along the two directions are either both normal or both anomalous.

The above results obviously generalize to the d-dimensional case. The fully-averaged displacement $\langle x_1^{(i)}(z) \rangle$ $(1 \leqslant i \leqslant d)$ has the exact expression

$$\langle x_1^{(i)}(z) \rangle = z^{-2} \frac{\langle A_i(z) \rangle}{R_d(z)}$$

where now

$$A_{i,nm}(z) = W_{i,nm} G_{d,nm}(z), \quad G_{d,nm}(z) = \frac{1}{z + \sum_{i=1}^{d} W_{i,nm}}, \quad R_d(z) = \langle G_d(z) \rangle$$

The anomalous region is now bounded by the hyperplane with equation

$$\mu \equiv \sum_{i=1}^{d} \mu_i = 1$$

Thus, as the space dimension is increased, it is harder and harder to encounter an anomalous drift, due to the fact that the above condition is more and more difficult to fulfill with many positive μ's. The same is true for a one-dimensional directed lattice with long-range jumps.[11] In a way, the size of the domain of existence of the anomalous phase tends to diminish when the dimensionality gets higher.

On the other hand, it is seen that, for a *directed* walk, anomalous phases for the velocity still exist in any space dimension; in that sense, the critical dimension is infinite. This is in sharp contrast with the results obtained for the both-way walk in RG methods with vanishing bias[4] or with bias in the presence of weak disorder as defined in ref. 9, establishing that $d=2$ is the lower critical dimensionality above which no anomalous dynamical phase is to be expected.

4. ASYMPTOTICS FOR THE MEAN SQUARE DISPLACEMENT

The determination of the dynamics for the mean square displacements is inevitably much more involved, but plainly proceeds through standard Laplace complex analysis. Essentially, in order to find the exponents for the disorder-averaged $\langle \delta_{uv}(t) \rangle$ as given by Eq. (9), one has to achieve an asymptotic analysis of the convolution $\langle u_1 * v_1 \rangle$ $(u, v = x, y)$ defined in the usual way:

$$(F * H)(z) = \int_C \frac{dz'}{2i\pi} F(z')\, H(z-z')$$

where C is a line in the z' plane which has the origin to the left and the fixed point z to the right. Obviously, if the quenched and annealed square displacements are *a priori* known to coincide, this difficult step is unnecessary and it is just sufficient to compute additional derivatives of the generating function $\langle \Phi \rangle$. One of our results is precisely the demonstration of this fact, thus establishing that the measure of the sample-to-sample fluctuations $\tilde{\sigma}_{xx}(t) = \langle \overline{x(t)}^2 \rangle - \langle \overline{x(t)} \rangle^2$ is indeed subdominant at large times as compared to $\delta_{xx,A}(t)$.

At first sight, the determination of quadratic quantities such as $\langle x_1(z)\, x_1(z') \rangle$ requires the knowledge of all the correlation functions $\langle P_{nm}(z)\, P_{n'm'}(z') \rangle$, which is rather hard to get. In order to achieve our goal, it is better, as we did in refs. 6 and 7, to use functional relations for the quantities $u_1(z)$ $(u = x, y)$ by considering two different starting points; these relations are established for the present two-dimensional case in Appendix A, taking advantage of the order relation between successively visited sites in a directed walk. The subsequent procedure will be given below in some detail for $\langle x_1(z)\, x_1(z') \rangle$, and we will only quote the results for the two other averages $\langle x_1(z)\, y_1(z') \rangle$ and $\langle y_1(z)\, y_1(z') \rangle$.

As compared to the one-dimensional case, the main difficulty here lies in the fact that there are many different correlated walks going from one point to another. The calculation of the above averages is thus not simply a rather straightforward extension of the $d=1$ results, but on the contrary requires first the knowledge of new quantities, as explained in Appendix B. The basic result obtained there is the following:

$$\langle x_1(z)\, x_1(z')\rangle = \frac{1}{zz'} \frac{\gamma_0 + 4\beta_0\gamma/[\alpha - 2\beta + (\alpha^2 - 4\beta^2)^{1/2}]}{\alpha_0 - (\beta_0/\beta)[\alpha - (\alpha^2 - 4\beta^2)^{1/2}]} \tag{21}$$

where α_0, α,... are various functions of (z, z') defined in Appendix B.

As ordinarily in the inverse Laplace transform, the point is now to elucidate the small-z behavior of the integral over z' of expression (21), after the substitution $z \to z', z' \to z - z'$. It turns out that the centered deviation

$$\sigma_{xx}(z', z - z') = \langle x_1(z')\, x_1(z - z')\rangle - \langle x_1(z')\rangle \langle x_1(z - z')\rangle \tag{22}$$

is a bit easier to analyze and it is sufficient to show whether the asymptotic behavior of this quantity is subdominant as compared to $\delta_{xx,A}(t)$ as previously defined. For future analysis, we first give the behavior of this latter quantity at large times. The Laplace transform of $\langle x_2(z)\rangle$ can be straightforwardly extracted from the generating function $\langle \Phi \rangle$. We find

$$\langle x_2(z)\rangle = z^{-2} \frac{\langle A_x(z)\rangle}{R(z)} + 2z^{-3} \frac{\langle A_x(z)\rangle^2}{R^2(z)}$$

By using the expansion for $R(z)$ [see Eq. (17)] and by taking the appropriate inverse transforms, it is readily seen that the annealed mean square displacement $\delta_{xx,A}(t)$ has the following asymptotic behaviors:

$$\mu < 1: \quad \delta_{xx,A}(t) = \left[\frac{\mu_x}{\mu} \frac{\sin \pi\mu}{\pi\mu}\right]^2 \left[2 \frac{[\Gamma(\mu+1)]^2}{\Gamma(2\mu+1)} - 1\right] (W_c t)^{2\mu} \tag{23}$$

$$1 < \mu < 2: \quad \delta_{xx,A}(t) = 2\left[\frac{\mu_x}{\mu}\right]^2 \frac{(\mu-1)^3}{(3-\mu)(2-\mu)} (W_c t)^{3-\mu} \tag{24}$$

$$2 < \mu: \quad \delta_{xx,A}(t) = \frac{\mu_x}{\mu} \frac{\mu-1}{\mu-2} \left[\mu + 2\left(\frac{\mu_x}{\mu} - 1\right)\right] W_c t \tag{25}$$

The small-z analysis of the integral over z' of $\sigma_{xx}(z', z - z')$ is rather tedious, all the more since the denominator in Eq. (21) contains a multivalued function (the square root) with a multivalued argument [all the functions α, β,... involve the multivalued function $R(z)$ defined in Eq. (14)]. We will only briefly sketch the full calculation, which indeed demonstrates that the convolution integral

$$\Sigma_{xx}(z) = \int_C \frac{dz'}{2i\pi} \sigma_{xx}(z', z - z') \tag{26}$$

eventually provides, at large times, subdominant contributions as compared to $\delta_{xx,A}(t)$ as given by Eqs. (23)–(25). Recall that $\Sigma_{xx}(z)$ is the Laplace transform of the sample-to-sample fluctuations of the thermal averaged coordinate, as measured by the mean square deviation $\tilde{\sigma}_{xx}(t)$, which precisely represents the difference between annealed and quenched diffusion coefficients.

The method proceeds as follows. One first determines all the singularities in the integrand of (26) which go to zero as z goes to zero. We did not found poles having this property; the only relevant singularities thus turn out to be the branching points of the various multivalued functions involved in the above expression. This being established, the integration contour is deformed so as to express the convolution integral in terms of real integrals corresponding to the two sides of the cuts. Then, a scaling of the integration variable allows one to obtain the z dependence of the integral in the small-z limit. A final Laplace inversion eventually provides the desired time dependence. Some more details are provided in the Appendix C for the less involved case, namely $1<\mu<2$.

As ordinarily, the various regimes arise because of the changing dominant behavior of the function $R(z)$ near $z=0$ as μ increases, as displayed by Eq. (17). For clarity, let us successively look at the different cases.

(a) $0<\mu<1$. In this case, $R(z)$ is dominated by the $z^{\mu-1}$ multivalued term. In addition, the square root in the denominator of σ_{xx} has no branching point going to zero with z, since its argument has no zero having this property, and the same is true for the poles of the integrand. For the asymptotic analysis, it is thus possible to move the integration contour to the curve shown on the left-hand part of Fig. 2.

It is subsequently seen that $\Sigma_{xx}(z)$ is dominated by a term $\text{const} \times z^{-(1+3\mu/2)}$ (where the constant prefactor can be calculated),

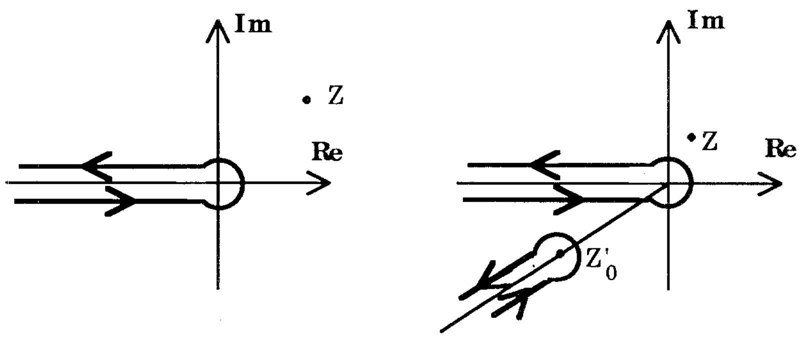

Fig. 2. Contours used in the convolution integral (30) for the two cases $\mu<2$ (left) and $\mu>2$ (right).

implying that the relevant deviation $\tilde{\sigma}_{xx}(t)$ behaves as $t^{3\mu/2}$ at large times, which is indeed negligible as compared to $\delta_{xx,A}(t)$. Thus, for $\mu < 1$, the two quenched and annealed dispersions coincide at large times and $\delta_{xx,A}(t) \sim \delta_{xx,Q}(t) \sim t^{2\mu}$ in the asymptotic regime.

(b) $1 < \mu < 2$. Now, $R(z)$ behaves as a constant near $z = 0$, but the subdominant $z^{\mu-1}$ multivalued term is still to be considered. As for the various singularities of the integrand, the situation is the same as above and it turns out that $\tilde{\sigma}_{xx}(t) \sim t^{5/2-\mu}$, which is again negligible as compared to $\delta_{xx,A}(t)$, which, in this case, behaves like $t^{3-\mu}$ [see Eq. (24)]; thus, one again obtains $\delta_{xx,A}(t) \sim \delta_{xx,Q}(t) \sim t^{3-\mu}$ at large times.

(c) $2 < \mu$. The main quantitative difference from the previous cases is that now the square root in the denominator of σ_{xx} [see Eqs. (21) and (22)] vanishes when z goes to zero. In terms of reduced variables Z and Z', the relevant zero Z'_0 is given by

$$Z'_0 = -\left(\frac{\mu}{2} - 1\right)^{1/2} Z^{1/2} + \frac{Z}{2} + \cdots$$

The relevant contour for the asymptotic analysis can thus be chosen as shown in the right-hand part of Fig. 2. The dominant term for Σ_{xx} yields $\tilde{\sigma}_{xx}(t) \sim t^{1/2}$, which again is negligible as compared to t [see Eq. (25)]. Thus, $\delta_{xx,A}(t) \sim \delta_{xx,Q}(t) \sim t$ for $\mu > 2$.

We are now in the position to state that, for any μ, the quenched and annealed mean square displacements coincide in the asymptotic regime for any μ:

$$\delta_{xx,Q} \sim \delta_{xx,A}, \qquad t \to +\infty, \quad \forall \mu \tag{27}$$

Thus, by setting

$$\delta_{xx,Q}(t) = 2D_{xx}(\mu)(W_c t)^{\beta(\mu)} \tag{28}$$

and using Eqs. (23)–(25), one can collect the results giving the final behavior of $\delta_{xx,Q}(t)$, displayed in the last column of Table I. The dominant exponent for $\delta_{xx,A}(t)$, $\beta(\mu)$, and the dominant exponent $\gamma(\mu)$ for the difference $\delta_{xx,A}(t) - \delta_{xx,Q}(t)$ are plotted in Fig. 3 as functions of μ; it is interesting to note that, as far as a normal drift occurs ($\mu > 1$), the relative importance of disorder fluctuations behaves like $t^{-1/2}$. The μ dependence of $D_{xx}(\mu)$ is illustrated in Fig. 4.

The results for $\delta_{yy,Q}(t)$ are found just by exchanging μ_x and μ_y in the above expressions. As for the crossed term $\delta_{xy,Q}(t)$, one has to replace γ and γ_0 by new expressions $\tilde{\gamma}_0$ and $\tilde{\gamma}$ in the linear system (B2)–(B3); this

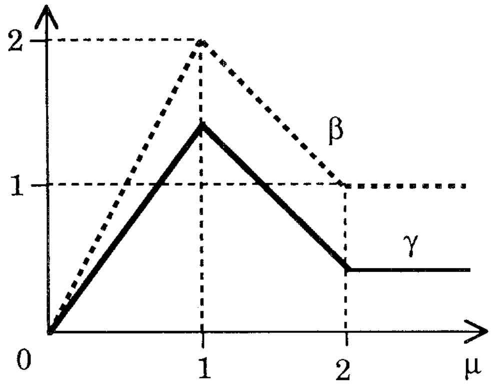

Fig. 3. Dominant exponents for $\delta_{xx,Q}(t)$ (β, dashed line) and for the difference $\delta_{xx,A}(t)-\delta_{xx,Q}(t)$ (γ, solid line).

does not affect in any way the conclusions about the subdominant character of the difference between $\delta_{xy,Q}(t)$ and $\delta_{xy,A}(t)$. The crossed correlation $\delta_{xy,Q}(t)$ is thus easily explicitly found from the averaged generating function $\langle \Phi \rangle$. We find

$$\mu<1: \quad \delta_{xy,Q}(t)=\frac{\mu_x\mu_y}{\mu^2}\left[\frac{\sin \pi\mu}{\pi\mu}\right]^2\left[2\,\frac{[\Gamma(\mu+1)]^2}{\Gamma(2\mu+1)}-1\right](W_c t)^{2\mu} \tag{29}$$

$$1<\mu<2: \quad \delta_{xy,Q}(t)=2\,\frac{\mu_x\mu_y}{\mu^2}\,\frac{(\mu-1)^3}{(3-\mu)(2-\mu)}\,(W_c t)^{3-\mu} \tag{30}$$

$$2<\mu: \quad \delta_{xy,Q}(t)=2\,\frac{\mu_x\mu_y}{\mu^2}\,\frac{\mu-1}{\mu-2}\,W_c t \tag{31}$$

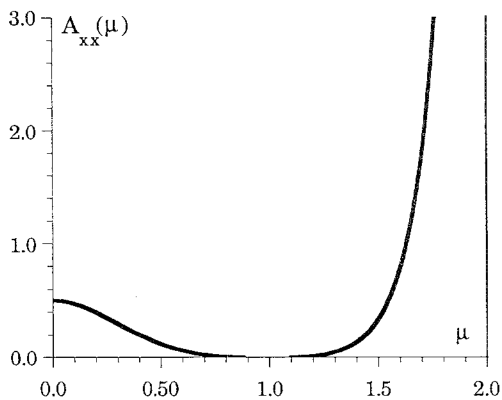

Fig. 4. Variations as a function of μ of the coefficient $A_{xx}(\mu)$ defined as $A_{xx}(\mu)=(\mu/\mu_x)^2\,D_{xx}(\mu)$.

Note that, by forming the quantity

$$\Xi = \langle \overline{x^2} \rangle + \langle \overline{y^2} \rangle + 2\langle \overline{xy} \rangle - \langle \bar{x} \rangle^2 - \langle \bar{y} \rangle^2 - 2\langle \bar{x} \rangle \langle \bar{y} \rangle$$

one recovers the result for the Bethe lattice with the coordinance $Z = d + 1 = 3$[12]; in this last paper, it was also shown that the quenched and the annealed quantities indeed asymptotically coincide for a Bethe lattice.

It is nearly evident that the identity between the quenched and annealed dispersions just demonstrated for $d = 2$ also holds true for higher dimension. The various (normal or anomalous) regimes obtained in this section for $d = 2$ thus generalize to any d in the same way as is done at the end of Section 3 for the first moments of the position. We thus reach the conclusion that anomalous phases still exist for a directed walk in any spatial dimension d and for strong disorder, the strength of it being measured by the parameter μ.

5. CONCLUSIONS

The dynamical exponents for the coordinate and for the mean square displacement have been calculated for a two-dimensional random-random directed walk and turn out to be the same as for the one-dimensional model, with a proper redefinition of the parameter μ. For the velocity as well as for the mean square deviations, the results can be easily extended to an arbitrary dimension d and it is seen that an anomalous region still exists for any d, although its size in the parameter space keeps diminishing as the space dimension increases. In other words, anomalous phases are always present, provided the surrounding disorder is strong enough.

Our results about the dynamical exponents display the fact that the present model seems to capture only a part of the two-dimensional general walk. On one hand, the self-averaging properties gained by an increase of space dimensionality are already present in the directed walk, since, as shown above, the quenched and annealed diffusion coefficients indeed coincide in the asymptotic regime, normal or anomalous. On the other hand, anomalous phases always exist in the directed model; on the contrary, RG methods lead to the conclusion that, for the general walk with vanishing bias or with a finite bias and weak local random perturbations, no anomalous phase is expected in a space with a dimension greater than 2. The status of the equivalence on large time and space scales between the directed and the both-way walks, which is believed to be true for $d = 1$, is thus unclear for higher dimension. This point deserves further study.

APPENDIX A

In order to establish the functional relation for the coordinate, we first note that, using the master equation (3), we can rewrite the velocity along the x axis, $zx_1(z)$, as

$$zx_1(z) = \sum_{n,m=0}^{+\infty} W_{x,nm} P_{nm}(z) \tag{A1}$$

We now need a temporarily slightly more complicated notation. Let $P_{nm}^{(pq)}(z)$ denote the Laplace transform of the probability to be at (n, m) when the starting point was the site located at (p, q). Note that the $P_{nm}^{(pq)}(z)$ satisfy Eq. (12) with $P_{pq}^{(pq)}(z) = G_{pq}(z)$. The corresponding velocity along x is then given by

$$zx_1^{(pq)}(z) = \sum_{n,m=0}^{+\infty} W_{x,nm} P_{nm}^{(pq)}(z) \tag{A2}$$

With $(p, q) = (0, 0)$, this last series can be decomposed as follows:

$$W_{x,00} P_{00}^{(00)}(z) + \sum_{n=1}^{+\infty} W_{x,n0} P_{n0}^{(00)}(z) + \sum_{m=1}^{+\infty} W_{x,0m} P_{0m}^{(00)}(z) + \sum_{n=1}^{+\infty} \sum_{m=1}^{+\infty} W_{x,nm} P_{nm}^{(00)}(z) \tag{A3}$$

Clearly one has

$$W_{x,n0} P_{n0}^{(00)}(z) = \prod_{n'=0}^{n'=n} W_{x,n'0} G_{n'0}(z) = W_{x,00} G_{00}(z)\, W_{x,n0} P_{n0}^{(10)}(z)$$

Thus, the second and third terms in Eq. (A2) can be written respectively as

$$W_{x,00} G_{00}(z) \sum_{n=1}^{+\infty} W_{x,n0} P_{n0}^{(10)}(z) \quad \text{and} \quad W_{y,00} G_{00}(z) \sum_{m=1}^{+\infty} W_{x,0m} P_{0m}^{(01)}(z)$$

We now transform the last term in Eq. (A3) by noting that $P_{nm}^{(00)}$ is itself the sum over many paths, which can be divided into two classes: (i) those which all cross the (1, 0) site, and (ii) those which all cross the (0, 1) site.

The paths of the first class give terms which all have the common factor $W_{x,00} G_{00}$. In the same way, all the terms arising from the second class have the common factor $W_{y,00} G_{00}$. Once the factorization is achieved, one can write

$$P_{nm}^{(00)}(z) = W_{x,00} G_{00}(z) \sum_{\text{all paths crossing } (1,0)} + W_{y,00} G_{00}(z) \sum_{\text{all paths crossing } (0,1)}$$

The two last summations are nothing else than $P^{10}_{nm}(z)$ and $P^{01}_{nm}(z)$, respectively. Finally, by collecting terms, we find

$$zx_1^{(00)}(z) = W_{x,00} G_{00}(z) + z W_{x,00} G_{00}(z)\, x_1^{(10)}(z) + z W_{y,00} G_{00}(z)\, x_1^{(01)}(z) \quad \text{(A4)}$$

This relation can be generalized as

$$\begin{aligned} zx_1^{(nm)}(z) = {} & W_{x,nm} G_{nm}(z) + z W_{x,nm} G_{nm}(z)\, x_1^{(n+1m)} \\ & + z W_{y,nm} G_{nm}(z)\, x_1^{(nm+1)}(z) \end{aligned} \quad \text{(A5)}$$

In the same way, one finds

$$\begin{aligned} zy_1^{(nm)}(z) = {} & W_{y,nm} G_{nm}(z) + z W_{y,nm} G_{nm}(z)\; y_1^{(nm+1)}(z) \\ & + z W_{x,nm} G_{nm}(z)\; y_1^{(m+1n)}(z) \end{aligned} \quad \text{(A6)}$$

These functional relations obviously reduce to the one given in refs. 6 and 7 for the one-dimensional case, and in ref. 12 for the Bethe lattice. In addition, they can also be used to derive the disorder average given in Eq. (16) of the present paper.

APPENDIX B

Let us now define the following correlation functions for the velocities along the x axis for the two different starting points $(n, 0)$ and $(0, n)$:

$$C_n(z, z') = zz' \langle x_1^{(n0)}(z)\, x_1^{(0n)}(z') \rangle \qquad (n \geq 0) \quad \text{(B1)}$$

We are essentially interested in finding $C_0(z, z')$, but, as clearly seen from the functional relations established in Appendix A, each C_n is coupled with $C_{n\pm 1}$. The coupled equations are set up by first using twice the functional relation (A5) for $x_1^{(n0)}(z)$ and for $x_1^{(0n)}(z')$. The two corresponding relations are then multiplied term by term and the disorder average taken. Using the translational symmetry for the averages, we obtain the following linear system for the C_n:

$$\alpha_0(z, z')\, C_0(z, z') - \beta_0(z, z')[C_1(z, z') + C_1(z', z)] = \gamma_0(z, z') \quad \text{(B2)}$$

$$\begin{aligned} & -\beta(z, z')\, C_{n-1}(z, z') + \alpha(z, z')\, C_n(z, z') - \beta(z, z')\, C_{n+1}(z, z') \\ & \quad = \gamma(z, z') \qquad (n \geq 1) \end{aligned} \quad \text{(B3)}$$

In the last equations, the various coefficients are equal to

$$\alpha_0(z, z') = 1 - \langle W_x^2 GG' \rangle - \langle W_y^2 GG' \rangle$$

$$\beta_0(z, z') = \langle W_x W_y GG' \rangle$$

$$\gamma_0(z, z') = \langle W_x^2 GG' \rangle + \langle W_x WGG' \rangle \left[\frac{\langle W_x G' \rangle}{z'R'} + \frac{\langle W_x G \rangle}{zR} \right]$$

$$\alpha(z, z') = 1 - \langle W_x G \rangle \langle W_x G' \rangle - \langle W_y G \rangle \langle W_y G' \rangle$$

$$\beta(z, z') = \langle W_x G \rangle \langle W_y G' \rangle$$

$$\gamma(z, z') = \langle W_x G \rangle \langle W_x G' \rangle + \langle W_x G \rangle \langle WG' \rangle \left[\frac{\langle W_x G' \rangle}{z'R'} + \frac{\langle W_x G \rangle}{zR} \right]$$

where, for short, unprimed symbols denote functions of the variable z, primed symbols functions of z' [$G = G(z)$, $G' = G(z')$, and so on]. All the above averages can be easily computed with the gamma distributions ρ_u as given in Eq. (4). Thus

$$\langle A_u(z) \rangle \equiv \langle W_u G(z) \rangle = \frac{\mu_u}{\mu} R(z)$$

$$\langle W_u^2 G(z)\, G(z') \rangle = \frac{\mu_u(\mu_u + 1)}{\mu(\mu + 1)} F(z, z')$$

$$\langle W_x W_y G(z)\, G(z') \rangle = \frac{\mu_x \mu_y}{\mu(\mu + 1)} F(z, z')$$

$$\langle W_u WG(z)\, G(z') \rangle = \frac{\mu_u}{\mu} F(z, z')$$

$$F(z, z') = 1 - \frac{z^2 R(z) - z'^2 R(z')}{z - z'}$$

It is seen that the above linear system (B2)–(B3) only admits the nontrivial symmetric solution

$$C_n(z, z') = C_n(z', z)$$

Indeed, the antisymmetric solution is readily seen to satisfy another linear homogeneous system, the determinant of which is nonvanishing identically.

The above infinite linear system (B2)–(B3) looks rather innocent, but must be carefully solved. This can be done at least in two ways:

(i) Either it is first truncated to a finite N-dimensional system; then the limit $N \to +\infty$ of the corresponding solution is taken.

(ii) Or, as one is essentially interested in finding $C_0(z, z')$, one expresses $C_n(z, z')$ as a function of $C_0(z, z')$ and one imposes on the former quantity that it not be diverging in the limit $n \to +\infty$. This condition provides an additional missing relation, allowing for a full complete solution.

Either method gives the same answer, namely

$$\langle x_1(z)\, x_1(z')\rangle = \frac{1}{zz'} \frac{\gamma_0 + 4\beta_0 \gamma/[\alpha - 2\beta + (\alpha^2 - 4\beta^2)^{1/2}]}{\alpha_0 - (\beta_0/\beta)[\alpha - (\alpha^2 - 4\beta^2)^{1/2}]} \tag{B4}$$

where, for short, the (z, z') dependence in the various functions has been omitted. The square-root function in the above expression is unambiguously defined by continuity from its real positive values.

As a by-product, it can be seen that the correlation function $C_n(z, z')$ has the limit

$$\lim_{n \to +\infty} C_n(z, z') = \langle x_1(z)\rangle \langle x_1(z')\rangle$$

This means that the correlations between the velocities corresponding to two different starting points indeed vanish when these are infinitely apart, a pleasant result on physical grounds. In addition, it is readily seen that the expression (B4) correctly reproduces the one-dimensional result, as it should, as well as the pure $d=2$ case.

APPENDIX C

We now will give some more details on the asymptotic analysis procedure used in Section 4, by examining the simplest case, namely $1 < \mu < 2$. For simplicity, we call N and D the numerator and the denominator of the expression for $\sigma_{xx}(z, z-z')$ [see Eq. (22)] which results from the use of Eqs. (16) and (21), keeping aside the prefactor $1/[z'(z-z')]$. By moving the contour, one has

$$\int_C \frac{dz'}{2i\pi} \frac{1}{z'(z-z')} = -\int_\varepsilon^{+\infty} \frac{dx}{2i\pi} \frac{1}{x(z+x)} \left[\frac{N_+}{D_+} - \frac{N_-}{D_-}\right] + I_\varepsilon \tag{C1}$$

In this equation, $N_\pm$ and $D_\pm$ denote the two values of the multivalued corresponding function, just above or just below the cut extending on the negative real axis. I_ε is the integral along a small circle with radius ε around the origin. Expression (B1) can be rewritten as

$$\int_C \frac{dz'}{2i\pi} \frac{1}{z'(z-z')} \frac{N}{D} = -\int_\varepsilon^{+\infty} \frac{dx}{\pi} \frac{1}{x(z+x)} \operatorname{Im}\left[\frac{N_-}{D_-}\right] + I_\varepsilon \equiv \mathscr{L} + I_\varepsilon \tag{C2}$$

The most divergent term in N_- is $(Z = z/W_c)$

$$-\left[\frac{\mu_x}{\mu}\right]^2 \frac{1}{2x+Z} \left\{\left[\frac{1}{\mu-1} - \Gamma(1-\mu)\, x^{\mu-1} e^{-i\mu\pi}\right]^{-1} - \left[\frac{1}{\mu-1} + \Gamma(1-\mu)(Z+x)^{\mu-1}\right]^{-1}\right\}$$

By scaling the integration variable x with Z, i.e., by putting $x = ZX$, this term becomes

$$-\left[\frac{\mu_x}{\mu}\right]^2 (\mu-1)^2 \Gamma(1-\mu) \frac{Z^{\mu-2}}{2X+1} [X^{\mu-1} e^{-i\mu\pi} + (1+X)^{\mu-1}]$$

With the same arguments, the most important term in the denominator D_- is seen to be

$$D_- = \frac{\mu}{\mu+1}\left[1 - \frac{(\Delta\mu)^2}{\mu^2}\right]^{1/2}\left[\frac{Z}{\mu-1}\right]^{1/2}$$

By collecting terms and taking the imaginary part, it is seen that the integral $\mathscr{L}$ is given by

$$-\left[\frac{\mu_x}{\mu}\right]^2 \frac{\mu+1}{\mu} \frac{\Gamma(1-\mu)(\mu-1)^{5/2} \sin \mu\pi}{[1-(\Delta\mu/\mu)^2]^{1/2}} Z^{\mu-7/2} \int_0^{+\infty} \frac{dX}{\pi} \frac{X^{\mu-2}}{(1+X)(2X+1)} \tag{C3}$$

The last integral is equal to $(1-2^{2-\mu})/\sin \mu\pi$. The I_ε term is easily shown to be, in the limit $\varepsilon \to 0$,

$$\left[\frac{\mu_x}{\mu}\right]^2 \frac{\mu+1}{\mu} \frac{\Gamma(1-\mu)(\mu-1)^{5/2}}{[1-(\Delta\mu/\mu)^2]^{1/2}} Z^{\mu-7/2} \tag{C4}$$

By taking Eqs. (C3) and (C4) into account, one obtains $\tilde{\sigma}_{xx} \sim \text{const} \times t^{(5/2)-\mu}$ as claimed in the main text.

REFERENCES

1. C. W. Gardiner, *Handbook of Stochastic Methods for Physics, Chemistry and the Natural Sciences* (Springer-Verlag, 1985).
2. S. Alexander, J. Bernasconi, W. R. Schneider, and R. Orbach, *Rev. Mod. Phys.* **53**:175 (1981).
3. B. Derrida, *J. Stat. Phys.* **31**:433 (1983).
4. J. P. Bouchaud and A. Georges, *Phys. Rep.* **195**:127 (1990).
5. P. Le Doussal, Thesis, Université Paris 6 (1987), unpublished.
6. C. Aslangul, M. Barthélémy, N. Pottier, and D. Saint-James, *J. Stat. Phys.* **59**:11 (1990).
7. C. Aslangul, M. Barthélémy, N. Pottier, and D. Saint-James, *J. Stat. Phys.* **61**:403 (1990).
8. C. Aslangul, M. Barthélémy, N. Pottier, and D. Saint-James, *Physica A* **171**:47 (1991).
9. J. Bricmont and A. Kupiainen, *Phys. Rev. Lett.* **66**:1689 (1991).
10. P. Le Doussal and J. Machta, *Phys. Rev. B* **40**:9427 (1989).
11. C. Aslangul, N. Pottier, and D. Saint-James, unpublished results.
12. C. Aslangul, M. Barthélémy, N. Pottier, and D. Saint-James, *J. Stat. Phys.*, to be published.